Nonlinear parametric quantization of gravity and cosmological models

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Abstract

A generalization of the recently formulated nonlinear quantization of a parameterized theory is presented in the context of quantum gravity. The parametric quantization of a Friedmann universe with a massless scalar field is then considered in terms of analytic solutions of the resulting evolution equations.

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In a recent paper [1] the quantization of a physical system with finite degrees of freedom subject to a Hamiltonian constraint was re-examined. A new scheme, called parametric quantization, was introduced in that work advocating the concept of treating a chosen time variable as a constrained classical variable coupled to the other dynamical variables to be quantized. The approach was motivated by the cosmological approach to quantum gravity [2] whose dynamical structure is analogous to that of the parameterized theory of a relativistic particle. In an $(n+1)$-dimensional pseudo-Riemannian spacetime with a metric tensor $\gamma_{\mu\nu}(q^\lambda)$ in coordinates q^μ, $(\mu, \nu = 0, 1, \cdots, n)$ the dynamics of such a particle subject to a potential $V(q^\lambda)$ is generated by a Lagrangian of the following generic form:

$$L(q^\mu, \dot{q}^\mu, N) = \frac{1}{2N}\gamma_{\mu\nu}\dot{q}^\mu\dot{q}^\nu - NV. \tag{1}$$

Here $q^\mu(t)$ describe the particle trajectory parameterized by t, whose choice is related to the function $N(t) > 0$, and $\dot{q}^\mu := \frac{\mathrm{d}\dot{q}^\mu}{\mathrm{d}t}$. Classically the particle's motion is governed by the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^\mu}\right) - \frac{\partial L}{\partial q^\mu} = 0 \tag{2}$$

subject to the constraint

$$\mathcal{H}(q^\mu, \dot{q}^\mu, N) := -\frac{\partial L}{\partial N} = \frac{1}{2N^2}\gamma_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + V = 0. \tag{3}$$

The equivalent canonical description is obtained from the "constrained Hamiltonian":

$$H(q^\mu, p_\mu, N) := \dot{q}^\mu p_\mu - L = N\mathcal{H} \tag{4}$$

in terms of the conjugate momenta $p_\mu := \frac{\partial L}{\partial \dot{q}^\mu} = \frac{1}{N}\gamma_{\mu\nu}\dot{q}^\nu$ where $\mathcal{H}$ can be re-express as

$$\mathcal{H}(q^\mu, p_\mu) = \frac{1}{2}\gamma^{\mu\nu}p_\mu p_\nu + V. \tag{5}$$

The canonical equations of motion are given by

$$\frac{\mathrm{d}q^\mu}{\mathrm{d}t} = \frac{\partial H}{\partial p_\mu}, \quad \frac{\mathrm{d}p_\mu}{\mathrm{d}t} = -\frac{\partial H}{\partial q^\mu} \tag{6}$$

subject to the Hamiltonian constraint

$$\mathcal{H}(q^\mu, p_\mu) = 0. \tag{7}$$

A quantization scheme may be set up by choosing one of the variables, say q^0 and its conjugate momentum p_0, as classical variables that interact with the remaining quantized variables q^a, $(a = 1, 2, \cdots, n)$ in a semi-classical fashion [1]:

$$1\frac{\partial \psi}{\partial t} = \hat{H}\psi \tag{8}$$

$$\frac{\mathrm{d}q^0}{\mathrm{d}t} = \left\langle \frac{\partial \hat{H}}{\partial p_0} \right\rangle, \quad \frac{\mathrm{d}p_0}{\mathrm{d}t} = -\left\langle \frac{\partial \hat{H}}{\partial q^0} \right\rangle \tag{9}$$

$$\left\langle \hat{\mathcal{H}} \right\rangle = 0 \tag{10}$$

where $\psi = \psi(q^a, t)$ and $\left\langle \hat{O} \right\rangle$ denotes the expectation value of any operator $\hat{O}$. [1] The operators $\hat{H}$ and $\hat{\mathcal{H}}$ are obtained by substituting $p_a \to \hat{p}_a := -1\frac{\partial}{\partial q^a}$ into H and $\mathcal{H}$ respectively (followed by a suitable factor ordering.) Equations (8), (9) and (10) can be cast into a more compact form that will prove to be advantageous in generalizing parametric quantization to a field theoretical framework. This reformulation is done by constructing an "unconstrained Hamiltonian":

$$h(q^a, p_a, q^0, \dot{q}^0, N) := \dot{q}^a p_a - L = H - \dot{q}^0\frac{\partial L}{\partial \dot{q}^0} \tag{11}$$

using a partial Legendre transformation of L by leaving out the "$\dot{q}^0 \to p_0$" transform as performed in (4). With this the classical motion obeys the following canonical-type equations

$$\frac{\mathrm{d}q^a}{\mathrm{d}t} = \frac{\partial h}{\partial p_a}, \quad \frac{\mathrm{d}p_a}{\mathrm{d}t} = -\frac{\partial h}{\partial q^a} \tag{12}$$

plus the constraint-like equation

$$\mathcal{H}(q^a, p_a, q^0, \dot{q}^0, N) = \frac{1}{N}\left(h + \dot{q}^0\frac{\partial L}{\partial \dot{q}^0}\right) = 0. \tag{13}$$

Accordingly, in terms of the operators $\hat{h}(q^a, \hat{p}_a, q^0, \dot{q}^0, N)$ and $\hat{\mathcal{H}}(q^a, \hat{p}_a, q^0, \dot{q}^0, N)$ obtained by substituting $p_a \to \hat{p}_a$ into h and $\mathcal{H}$ given in (11) and (13), the system of equations (8), (9) and (10) now becomes

$$1\frac{\partial \psi}{\partial t} = \hat{h}\psi \tag{14}$$

$$\left\langle \hat{\mathcal{H}} \right\rangle = 0. \tag{15}$$

[1]Units in which $c = \hbar = 16\pi G = 1$ are adopted throughout.

For any given positive $N(t)$, equations (14) and (15) constitute a system of evolution for the wavefunction $\psi(q^a,t)$ and classical variable $q^0(t)$ whose initial data may be specified arbitrarily. It is evident that the presence of (15) makes the quantum theory *nonlinear*. (See, for instance, [6, 7] for other examples of nonlinear quantum theories.)

This formulation suggests a parametric quantization of gravity whose formal treatment can be outlined as follows. Start from the standard ADM Lagrangian for general relativity [5]:

$$L[g_{ij},\dot{g}_{ij},N_\mu]:=\int N\sqrt{g}\left(K_{ij}K^{ij}-K^2+R\right)\mathrm{d}^3x \tag{16}$$

in terms of the 3-metric $g_{ij}(x)=g_{ij}(x^k,t)$, $g=\det(g_{ij})$, lapse function $N(x)=N_0(x)$, shift vector $N_i(x)$, extrinsic curvature K_{ij}, $K=g^{ij}K_{ij}$ and intrinsic scalar curvature R of the evolving 3-geometry. ($\mu=0,1,2,3; i,j,k=1,2,3.$) A natural way of isolating the "true" gravitational degrees of freedom is to transform from $g_{ij}(x)$ to a set of embedding variables: $\vartheta^\mu(x),\mu=0,1,2,3$, and unconstrained variables: $\varphi^r(x),r=1,2$ [3]. Here ϑ^0 specifies time slicing and ϑ^a sets spatial coordinate condition. We then have $L=L[\varphi^r,\varpi_r,\vartheta^\mu,\dot{\vartheta}^\mu,N_\mu]$. The idea now is to regard ϑ^μ as constrained classical variables coupled "semi-classically" to the quantized true degrees of freedom carried by φ^r.

In view of the discussions above we introduce the unconstrained Hamiltonian

$$h[\varphi^r,\varpi_r,\vartheta^\mu,\dot{\vartheta}^\mu,N]:=\int\dot{\varphi}^r\varpi_r\mathrm{d}^3x-L \tag{17}$$

with the conjugate momenta

$$\varpi_r(x):=\frac{\delta L}{\delta\dot{\varphi}^r(x)}. \tag{18}$$

Further, we may derive the 'super-Hamiltonian' $\mathcal{H}=\mathcal{H}^0$ and 'super-momenta' $\mathcal{H}^i$ as follows:

$$\mathcal{H}^\mu[\varphi^r,\varpi_r,\vartheta^\nu,\dot{\vartheta}^\nu,N_\nu;x):=\frac{\delta h}{\delta N_\mu(x)} \tag{19}$$

($\mu,\nu=0,1,2,3$; $i=1,2,3$; $r=1,2$.) By analogy with (14) and (15) the nonlinear parametric quantization of gravity may be formulated with the following system of equations:

$$\imath\frac{\partial\psi}{\partial t}=\hat{h}\psi \tag{20}$$

$$\left\langle\hat{\mathcal{H}}^\mu\right\rangle=0 \tag{21}$$

that *generates the nonlinear evolution of the quantum state* $\psi[\varphi^r;t)$ *and classical embedding variables* $\vartheta^\mu(x^k,t)$, using the operators $\hat{h}[\varphi^r,\hat{\varpi}_r,\vartheta^\nu,\dot{\vartheta}^\nu,N_\nu]$ and $\hat{\mathcal{H}}^\mu[\varphi^r,\hat{\varpi}_r,\vartheta^\nu,\dot{\vartheta}^\nu,N_\nu;x)$ with $\hat{\varpi}_r:=-\imath\frac{\delta}{\delta\varphi^r}$.

It is an ongoing research program to further investigate the quantization of gravity using (20) and (21) that effectively quantizes only two out of the six spatial metric components as expected. In the meantime, it is instructive to investigate the implication of the above quantization scheme via a quantum cosmological approach. Therefore the rest of this letter will be devoted to the parametric quantization of a Friedmann universe with a massless scalar field based on the much simpler set of equations (14) and (15). Such a cosmological model has the Lagrangian [1, 4]:

$$L(\phi,\dot{\phi},R,\dot{R},N)=-\frac{6R}{N}\dot{R}^2+\frac{R^3}{2N}\dot{\phi}^2+6NKR \tag{22}$$

in terms of the massless scaler field $\phi(t)$, scale factor $R(t)$ and lapse function $N(t)$ that enter into the Robertson-Walker metric

$$g=-N^2\mathrm{d}t^2+R^2\sigma. \tag{23}$$

Here, as usual, σ is the metric on the homogeneous and isotropic 3-space of constant curvature K, with $K = 1, 0, -1$ corresponding to the closed, flat and open cases respectively. Regarding R as the classical time variable we obtain from (11) and (13) the following expressions:

$$h = 6R\frac{\dot{R}^2}{N} + \frac{Np^2}{2R^3} - 6NKR \tag{24}$$

$$\mathcal{H} = -6R\frac{\dot{R}^2}{N^2} + \frac{p^2}{2R^3} - 6KR \tag{25}$$

where

$$p := \frac{\partial L}{\partial \dot{\phi}} = \frac{R^3}{N}\dot{\phi}. \tag{26}$$

Comparing with the generic Lagrangian in (1) for $n = 1$ we see that the current model has the nonzero metric components and potential as follows:

$$\gamma_{00} = -12R, \quad \gamma_{11} = R^3 \tag{27}$$

$$V = -6KR. \tag{28}$$

By substituting $p \to \hat{p} := -\imath\frac{\partial}{\partial\phi}$ into h the operators

$$\hat{h} = -\frac{N}{2R^3}\frac{\partial^2}{\partial\phi^2} + 6R\frac{\dot{R}^2}{N} - 6NKR \tag{29}$$

$$\hat{\mathcal{H}} = -\frac{1}{2R^3}\frac{\partial^2}{\partial\phi^2} - 6R\frac{\dot{R}^2}{N^2} - 6KR \tag{30}$$

are constructed that will act on a wavefunction $\psi(\phi, t)$ (of weight $\frac{1}{2}$ with respect to the 1-dimensional metric γ_{11}) which is normalized according to:

$$\langle\psi, \psi\rangle := \int_{-\infty}^{\infty} |\psi(\phi, t)|^2 \, \mathrm{d}\phi = 1. \tag{31}$$

On parametric quantization the evolution equations for the wavefunction ψ and classical variable $R(t)$ follow from (14) and (15) to be: [2]

$$\imath\frac{\partial\psi}{\partial t} = -\frac{N}{2R^3}\frac{\partial^2\psi}{\partial\phi^2} \tag{32}$$

$$-6R\frac{\dot{R}^2}{N^2} + \frac{P^2}{2R^3} - 6KR = 0 \tag{33}$$

where

$$P^2 := -\left\langle \frac{\partial^2}{\partial\phi^2} \right\rangle. \tag{34}$$

The quantum evolution of the present cosmological model is therefore described by the above system of *nonlinear integro-partial differential equations*. Nonetheless, because of the absence of the "potential" term (mass of the scalar being zero) it is possible to find analytic solutions by regarding (32) as the Schrödinger equation of a free non-relativistic particle with a time-varying mass. The analysis will be

[2]The term $6R\frac{\dot{R}^2}{N^2} - 6KR$ in $\hat{h}$ is dropped as it contributes only to an overall phase in ψ. However this term must be retained in applying (15).

done in the $N = R$ gauge in order to compare with classical solutions [4]. Thus (32) is solved first to yield

$$\psi(\phi, t) = \int_{-\infty}^{\infty} \frac{A(k)}{\sqrt{2\pi}} \, e^{\imath[k\phi - F(t)k^2]} \mathrm{d}k \tag{35}$$

where $A(k)$ is an arbitrary complex-valued function and

$$F(t) := \int_{t_0}^{t} \frac{\mathrm{d}t'}{2R(t')^2} \tag{36}$$

for a chosen reference time t_0. At this point we can see that

$$P^2 = \int_{-\infty}^{\infty} k^2 \, |A(k)|^2 \, \mathrm{d}k \tag{37}$$

which is a positive constant as anticipated with the free particle analogy. Feeding this into (33), with suitable choice of the origin of the coordinate time t, we immediately obtain $R(t)$ to be

$$R(t)^2 = \begin{cases} \frac{|P| \sin 2t}{2\sqrt{3}} & (K = 1) \\ \frac{|P| t}{\sqrt{3}} & (K = 0) \\ \frac{|P| \sinh 2t}{2\sqrt{3}} & (K = -1) \end{cases} . \tag{38}$$

Substituting these into (36) we find that

$$F(t) = \begin{cases} \frac{\sqrt{3}}{2|P|} \ln \frac{\tan t}{\tan t_0} & (K = 1) \\ \frac{\sqrt{3}}{2|P|} \ln \frac{t}{t_0} & (K = 0) \\ \frac{\sqrt{3}}{2|P|} \ln \frac{\tanh t}{\tanh t_0} & (K = -1) \end{cases} . \tag{39}$$

Solutions (38) are in fact same as in the classical case if P is replaced by the classical momentum p defined in (26) which is also a constant (of classical motion.) Unlike the classical case, though, it is possible to envisage a quantum state with zero mean scalar momentum ($\langle \hat{p} \rangle = 0$) but nonzero deviation of the scalar momentum ($\langle \hat{p}^2 \rangle = P^2 > 0$). In this case, the evolution of the Friedmann universe can be thought of as being "purely quantum-driven." We conclude this letter by providing an explicit example for such a scenario. If at $t = t_0$ the wavefunction is given by the normalized gaussian packet with a variance δ_0 of the form:

$$\psi(\phi, t_0) = (2\pi)^{-\frac{1}{4}} \delta_0{}^{-\frac{1}{2}} \, e^{-\frac{(\phi - \phi_0)^2}{4\delta_0^2}} \, e^{\imath k_0 (\phi - \phi_0)} \tag{40}$$

then

$$A(k) = (2\pi)^{-\frac{1}{4}} \sigma^{-\frac{1}{2}} \, e^{-\frac{(k - k_0)^2}{4\sigma^2}} \, e^{-\imath k \phi_0} \tag{41}$$

where k_0 is the mean and $\sigma = \frac{1}{2\delta_0}$ is the variance of the k-distribution. Substituting (41) into (35) we obtain the following:

$$\psi(\phi, t) = (2\pi)^{-\frac{1}{4}} \left(\frac{\delta_0^2 + \imath F(t)}{\delta_0} \right)^{-\frac{1}{2}} e^{-\frac{\phi - \phi_0 - 2k_0 F(t)}{4(\delta_0^4 + F(t)^2)/\delta_0^2}} \, e^{\imath \left[k_0 (\phi - \phi_0) + \frac{\phi - \phi_0 - 2k_0 F(t)}{4(\delta_0^4 + F(t)^2)/F(t)} - k_0^2 F(t) \right]} \tag{42}$$

This is a moving wave packet with the time-dependent standard deviation given by

$$\delta(t) := \left| \frac{\delta_0^2 + \imath F(t)}{\delta_0} \right| = \sqrt{\frac{\delta_0^4 + F(t)^2}{\delta_0^2}}. \tag{43}$$

Clearly this yields the probabilistic density

$$|\psi(\phi, t)|^2 = \frac{1}{\sqrt{2\pi}\delta(t)}\, e^{-\frac{(\phi-\phi_0-2k_0F(t))^2}{2\delta(t)^2}}. \tag{44}$$

Furthermore, from (34) we have

$$P^2 = k_0^2 + \sigma^2. \tag{45}$$

In particular, the choice of $x_0 = k_0 = 0$ gives rise to the "purely quantum-driven" cosmological evolution discussed above.

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